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The theory of angular momenta and higher SU(n) symmetries

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Abstract. We discuss interrelations between generating invariants (GI) for the reduction $\Pi_{i=1}^{m} D(P_i) \to D(\dot{O}_{n-1})$ of the m-fold tensor product of SU(n) irreducible representations (irreps) $D(P_i)$ and polynomial bases of the SU(m) irreps $D(\dot{O}_{n-1}J, \dot{O}_{m-n-1})$, $\dot{O}_k = 0, 0, \ldots, 0$ (k times). A realisation of the SU(m) irrep $D(OJ, \dot{O}_{m-3})$ bases is given in terms of GI for SU(2) coupling (Wigner) coefficients. As a byproduct an expression is obtained for SU(2) 6-j symbols in terms of only two Wigner coefficients. We also discuss some possibilities of the analysis involved in solving the Wigner-Biedenharn problem (construction of orthonormal sets of the Wigner coefficients) for SU(n) groups ($n \geq 3$).

1. Introduction

The groups of unitary symmetry U(n) and SU(n) are used widely in various branches of modern theoretical physics (see, e.g., Lichtenberg 1978, Wybourne 1970, Karassiov 1985 and references therein). To fruitfully use these symmetries in physical applications it is necessary to develop an adequate mathematical technique for different groups SU(n). Such a technique should include the construction of (i) bases of irreducible representations (irreps), (ii) Wigner-Racah algebras (Clebsch-Gordan (or Wigner) coefficients and their covariant combinations) and (iii) generalised coherent states (see, e.g., Ališauskas 1983, Butler 1975, Louck 1979, Karassiov and Shelepin 1980). However, these problems are solved completely enough only for the SU(2) group, and not for the case SU(n) with $n \geq 3$ (see, e.g., Butler 1975, Biedenharn and Flath 1984, Black *et al* 1983).

The generating invariant (GI) method, in the form proposed by Karassiov (1976), may become an efficient tool for a unified solution of these problems. This method, introduced in the theory of physical symmetries by Van der Waerden (1932) and Weyl (1931, 1939), has been used fruitfully by Schwinger (1952), Regge (1958) and Bargmann (1962) in the quantum theory of angular momenta (see also Biedenharn and Louck 1981).

Resnikoff (1967), Karassiov (1973), Karassiov and Shelepin (1968, 1980) and Karassiov *et al* (1979), amongst others, extended applications of the GI method to the formalism of higher groups SU(n), $n \geq 3$†. Specifically, Karassiov *et al* (1979) and Karassiov and Shelepin (1980) found normalised GI for multiplicity-free Wigner coefficients and developed an algebraic technique for constructing GI for arbitrary Wigner coefficients of SU(n) groups. Karassiov and Shelepin (1980) established a close connection between GI for the Wigner coefficients and generalised coherent states of SU(n) groups.

† The GI method was also applied for determining Clebsch-Gordan coefficients of other simple Lie groups (see, e.g., Hongoh *et al* 1974, Gaskell and Sharp 1982).

Developing some ideas of earlier works (Moshinsky 1963, Karassiov and Shelepin 1968), Karassiov (1985) pointed out that there is an intimate connection between the task of constructing orthonormalised sets of GI for the Wigner coefficients of some groups SU(n) and polynomial bases for special irreps of other groups SU(m). In the present paper we examine this question in more detail for the case $n=2$ and arbitrary m.

The paper is organised as follows. After a few preliminaries in § 2 we formulate the key proposition of our analysis concerning the interrelations betwen GI and irrep bases of different SU(n) groups. In § 3 we give a recursive technique for simultaneously constructing orthonormalised GI for the mth rank Wigner coefficients† of SU(2) and the corresponding bases of the SU(m) irrep $D(OJ\dot{O}_{m-3})$ ($\dot{a}_k \equiv a, \ldots, a$ (k times)) in terms of Van der Waerden GI for the Wigner coefficients of SU(2), and then we discuss some corollaries. In § 4 we consider some possibilities of extending the results obtained to solving the Wigner–Biedenharn problem (construction of orthonormalised sets of the Wigner coefficients) for higher SU(n) groups (see Wigner 1941, Biedenharn 1962).

2. General remarks

The starting point of our analysis is a realisation of an action of the group SU(mn) and of its subgroups SU(m) and SU(n) ($n \leq m$) on a rectangular matrix $\|x_i^\alpha\|_{i=1,2,\ldots,m}^{\alpha=1,2,\ldots,n}$ where the group SU(n) acts on the upper indices and the group SU(m) acts on the lower ones. The generators $E_{ij}^{\alpha\beta}$ of U(mn) $\supset$ SU(mn) are

$$E_{ij}^{\alpha\beta} = x_i^\alpha \bar{x}_j^\beta \qquad \bar{x}_i^\alpha \equiv \partial/\partial x_i^\alpha \tag{2.1}$$

and generators of all other groups in our study are linear combinations of $E_{ij}^{\alpha\beta}$; particularly, $\tilde{E}_{ij}^{\alpha\beta} = E_{ij}^{\alpha\beta} - (1/mn)\delta_{\alpha\beta}\delta_{ij}\sum_{i,\alpha} E_{ii}^{\alpha\alpha}$ are generators of SU(mn), $\tilde{E}^{\alpha\beta} = \sum_i \tilde{E}_{ii}^{\alpha\beta}$ are those of SU(n), etc. We also use a realisation of SU(k) irrep spaces $\mathscr{L}_{\mathrm{SU}(k)}^{D}$ ($k = mn, m, n$) by homogeneous polynomials $\mathscr{P}(\{x_i^\alpha\})$ in variables x_i^α and with the inner product

$$\langle \mathscr{P}(\{x_i^\alpha\}) | \mathscr{P}'(\{x_\gamma^\beta\})\rangle \equiv \mathscr{P}^*(\{\bar{x}_i^\alpha\})\mathscr{P}'(\{x_j^\beta\})|_{\{x_i=0\}} \tag{2.2}$$

where the asterisk * denotes the complex conjugation of coefficients in $\mathscr{P}(\{x_i\})$ (cf Klink 1983).

Consider the symmetric irrep $D(P) = D(nJ, \dot{O}_{mn-2})$ of SU(mn) where $P = (p_1, p_2, \ldots, p_{mn-1})$ is the signature of the irrep, $p_i = M_i - M_{i+1}$, M_i are components of the U(mn) irrep $D[M_1, \ldots, M_{mn}]$ highest weight. From the spectral analysis of $D(nJ, \dot{O}_{mn-2})$ under the reduction SU(mn) $\supset$ SU(m) $\times$ SU(n) (see, e.g., Zhelobenko 1973) and complementarity of SU(m) and SU(n) actions on the space $\mathscr{L}_{\mathrm{SU}(mn)}^{D(nJ,\dot{O}_{mn-2})}$ of the irrep $D(nJ, \dot{O}_{mn-2})$ (see, e.g., Quesne 1973, Howe 1976) we find that the subspace $\mathscr{L}_{\mathrm{SU}(m)}^{D(\dot{O}_{n-1}J\dot{O}_{m-n-1})} \subset \mathscr{L}_{\mathrm{SU}(mn)}^{D(nJ,\dot{O}_{mn-2})}$ in which the irrep $D(\dot{O}_{n-1}J\dot{O}_{m-n-1})$ of SU(m) (and irrep $D[\dot{J}_n, \dot{O}_{m-n}]$ of U(m) $\supset$ SU(m)) acts, is spanned by SU(n)-invariant homogeneous polynomials $\mathscr{F}(\{x_1, x_2, \ldots, x_m\})$ composed of the m vectors $x_i = (x_i^\alpha)$, $i = 1, 2, \ldots, m$,

$$\mathscr{F}(\{x_i\}) = \sum_{J=\sum A_{i_1\ldots i_n}} C(\{A_{i_1\ldots i_n}\}) \prod_{1\leq i_1<i_2<\ldots<i_n\leq m} [x_{i_1}\ldots x_{i_n}]^{A_{i_1\ldots i_n}} \tag{2.3}$$

† These coefficients correspond to the reduction $\Pi_{i=1}^m D(2j_i) \to D(0)$ of the m-fold tensor product of SU(2) irreps $D(2j_i)$ (see Karassiov 1973).

where

$$[x_{i_1} \dots x_{i_n}] \equiv \varepsilon_{\alpha_1 \dots \alpha_n} x_{i_1}^{\alpha_1} \dots x_{i_n}^{\alpha_n} = \begin{vmatrix} x_{i_1}^1 & \dots & x_{i_n}^1 \\ x_{i_1}^2 & \dots & x_{i_n}^2 \\ \vdots & & \vdots \\ x_{i_1}^n & \dots & x_{i_n}^n \end{vmatrix} \tag{2.4}$$

is simultaneously an elementary invariant (component determinant) of SU(n) and a minor of component determinant of SU(m) (see Weyl 1939). In particular, the space $\mathscr{L}_{SU(m)}^{(\dot{O}_{n-1}J\dot{O}_{m-n-1})}$ contains, as its basic vectors, the GI $\mathscr{F}_{\{P_i\}}(\{x_i\})$ for the mth rank Wigner coefficients of SU(n) which correspond to the reduction

$$D(P_1) \times D(P_2) \times \dots \times D(P_m) \to D(\dot{O}_{n-1}) \qquad P_i = (p_i, \dot{O}_{n-2}) \tag{2.5}$$

with the constraints

$$\sum_{i=1}^{m} p_i = nJ \tag{2.6a}$$

$$E_{ii}\mathscr{F}_{\{P_i\}}(\{x_j\}) = p_i \mathscr{F}_{\{P_i\}}(\{x_j\}) \qquad i = 1, 2, \dots, m. \tag{2.6b}$$

(Note that (2.6b) specifies both the signatures of the irreps $D(p_i\dot{O}_{n-2})$ of SU(n) and the weights of GI $\mathscr{F}_{\{P_i\}}(\dots)$ as vectors of $\mathscr{L}_{SU(m)}^{(\dot{O}_{n-1}J\dot{O}_{m-n-1})}$.)

Specifically, the set $\{\mathscr{F}_{\{A_{i_1}\dots A_{i_n}\}}(\{x_i\}) = \Pi_{1 \leqslant i_1 < \dots} [x_{i_1} \dots x_{i_n}]^{A_{i_1} \dots i_n} : \Sigma_{\{i_j < i_{j+1}\}} A_{i_1 \dots i_n} = J\}$ of SU(n) invariants in (2.3) forms both the 'symmetric basis' (the extended Weyl basis) (see Karassiov 1973) of $D(\dot{O}_{n-1}J\dot{O}_{m-n-1})$ and the GI of the mth rank Wigner coefficients for the 'symmetric' coupling scheme of irreps $D(P_i)$ in (2.5) which was investigated by Kumar (1966) in the case $n = 2$ (we refer to those as the Kumar GI and coefficients).

The set $\{\mathscr{F}_{\{A_{i_1} \dots i_n\}}(\{x_i\})\}$ is complete (and even overcomplete) but not all the GI $\mathscr{F}_{\{A_{i_1}\dots A_{i_n}\}}(\{x_i\})$ are linearly independent and orthogonal. In physical applications orthonormal sets of both GI and basic vectors of irreps are important.

In order to define them we have to add some conditions to (2.6) which would fix the coupling scheme of SU(n) IR in (2.5) or a subgroup chain

$$U(m) \supset SU(m) \supset G_1 \supset G_2 \supset \dots G_i \supset \dots \tag{2.7}$$

as well as supplementary quantum numbers which are eigenvalues of some mutually commuting Hermitian operators and label orthonormal GI for the SU(n) Wigner coefficients or basic vectors of the irrep $D(\dot{O}_{n-1}J\dot{O}_{m-n-1})$ (or $D[\dot{J}_n\dot{O}_{m-n}]$). From the complementarity of the groups SU(n) and SU(m) on the space $\mathscr{L}_{SU(mn)}^{D(nJ,\dot{O}_{mn-2})}$ the key statement of our analysis follows. (i) There is one-to-one correspondence between the types of orthonormal polynomial bases of irreps $D(\dot{O}_{n-1}J\dot{O}_{m-n-1})$ of SU(m) (or $D[\dot{J}_n\dot{O}_{m-n}]$ of U(m)) and orthonormal sets of GI for the Wigner coefficients corresponding to (2.5), i.e. any choice of (2.7) simultaneously fixes the coupling scheme of irreps in (2.5), and vice versa. (ii) Every polynomial realisation of orthonormal basis of the irrep $D(\dot{O}_{n-1}J\dot{O}_{m-n-1})$, corresponding to a subgroup chain (2.7), yields simultaneously an orthonormal set of GI of SU(n) for the corresponding coupling scheme in (2.5), with the same set of labelling numbers, and vice versa.

For example, the polynomial canonical Gel'fand-Tsetlin basis of the irrep $D[\dot{J}_n\dot{O}_{m-n}]$ defined by the chain

$$U(m) \supset U(m-1) \supset \dots \supset U(3) \supset U(2) \supset U(1) \tag{2.8}$$

produces the set of GI which are defined by the following successive coupling scheme in (2.5):

$$(\ldots(((D(p_1\dot{O}_{n-2})\times D(p_2\dot{O}_{n-2}))\to D(p_1+p_2-2q,q,\dot{O}_{n-3}))\times D(p_3\dot{O}_{n-2})\to\ldots)\times\ldots$$
$$\times D(p_m\dot{O}_{n-2}))\to D(\dot{O}_{n-1}). \tag{2.9}$$

The other standard coupling schemes in (2.5), when only two irreps of SU(n) couple into an intermediate irrep at any stage, correspond to non-canonical types of bases of irrep of U(m) with the subgroup chains (2.7) chosen as

$$\begin{matrix} U(1)\subset U(2) & \\ \times & \subset U(4)\subset U(5)\subset\ldots\subset U(m) \\ U(1)\subset U(2) & \end{matrix} \tag{2.10a}$$

$$\begin{matrix} U(1)\subset U(2) & & \\ \times & \subset U(4) & \\ U(1)\subset U(2) & \times & \subset U(6)\subset\ldots\subset U(m) \\ & U(1)\subset U(2) & \end{matrix} \tag{2.10b}$$

$$\begin{matrix} \ldots\subset U(m_1) & \\ \times & \subset U(m) \qquad m=m_1+m_2 \\ \ldots\subset U(m_2) & \end{matrix} \tag{2.10c}$$

with the presence of the links $U(m_1)\times U(m_2)\times U(m_3)\times\ldots\times U(m_k)\subset U(m_1+m_2+\ldots+m_k)$ in (2.7), i.e. with the chain containing the toroidal subgroup $[U(1)]^{\otimes m}$ in the final step of the reduction.

The close connection obtained between GI for the SU(n) Wigner coefficients and polynomial bases of the irrep $D(\dot{O}_{n-1}J\dot{O}_{m-n-1})$ allows us to develop the Wigner-Racah algebras of SU(n) and the theory of bases of the SU(m) irreps $D(\dot{O}_{n-1}J\dot{O}_{m-n-1})$ simultaneously. Thereby any results found within one theory may be interpreted in terms of the other. We now discuss this point in more detail for the case $n=2$.

3. Theory of angular momenta from the viewpoint of the higher SU(m) symmetries

The orthonormal GI $\mathscr{F}_{\{j_1j_2j_3\}}(\{x_i\})$

$$\mathscr{F}_{\{j_i\}}(\{x_i\})\equiv\sum_{\{m_i\}}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix}\prod_{i=1}^{3}(x_i^1)^{j_i+m_i}(x_i^2)^{j_i-m_i}/[(j_i+m_i)!(j_i-m_i)!]^{1/2}$$
$$=\rho(j_1j_2j_3)[x_1x_2]^{J-2j_3}[x_1x_3]^{J-2j_2}[x_2x_3]^{J-2j_1} \qquad J=j_1+j_2+j_3 \tag{3.1}$$

(where $\rho(j_1j_2j_3)=[(J+1)!\Pi_{i=1}^{3}(J-2j_i)!]^{-1/2}(-1)^{J-2j_2}$ is a normalisation factor in accordance with (2.2) and j_i are coupled angular momenta), for the third rank Wigner coefficients† of SU(2) were known long ago (Van der Waerden 1932). Schwinger (1952) gave a method for constructing GI for Wigner coefficients of any rank (with standard coupling schemes of irreps), and Karassiov (1976) proposed a simpler technique for this purpose.

† These coefficients are basic quantities for constructing the Wigner-Racah algebras of SU(n) (see, e.g., Biedenharn and Louck 1981, Karassiov 1973).

Within our method the construction of GI for mth rank Wigner coefficients reproduces elementary operations of the SU(2) Wigner-Racah algebra and reduces to the action of operators of the type of $\mathscr{F}_{\{j_i\}}(\{\bar{x}_1, \bar{x}_2, \bar{x}_3\})$, both with $j_3 \neq 0$ and $j_3 = 0$, or operators $[(x_i\bar{x}_k) \equiv \Sigma_\alpha x_i^\alpha \bar{x}_k^\alpha]^s$ on the product of GI of the type of (3.1). As a result we obtain sets of orthonormal GI $\mathscr{F}_{\{j_i\}}^{A,\alpha}(\{x_1, x_2, \ldots, x_m\})$ for the mth rank Wigner coefficients which are labelled by the sets $\{j_i\}$, $\alpha = \{j_{\text{inter}}\}$ of coupled (j_i) and intermediate $\alpha = (j_{\text{inter}})$ momenta (the symbol A indicates the coupling scheme) and represented by quasimonomials with respect to $[\bar{x}_i, \bar{x}_j]$, $[x_m x_n]$ ('concise' form) or as weighted sums of the Kumar GI ('expanded' form). Consider some examples.

So, according to the algorithm, the GI $\mathscr{F}_{\{j_i\}}^{((12)(34)),j_{12}}(\{x_i\})$ for fourth rank Wigner coefficients (with the coupling scheme ((12)(34)) has the following concise form:

$$
\begin{aligned}
\mathscr{F}_{\{j_i\}}^{j_{12}}(\{x_i\}) &= \mathscr{F}_{\{j_i\}}^{((12)(34)),j_{12}}(\{x_1, \ldots, x_4\}) \\
&\equiv \sum_{\{m_i\}} \begin{pmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & -m_{12} \end{pmatrix} \begin{pmatrix} j_{12} & j_3 & j_4 \\ m_{12} & m_3 & m_4 \end{pmatrix} [2j_{12}+1]^{1/2} \\
&\quad \times (-1)^{j_1 - j_2 + m_{12}} \prod_{i=1}^{4} (x_i^1)^{j_i + m_i} (x_i^2)^{j_i - m_i} / [(j_i + m_i)!(j_i - m_i)!]^{1/2} \\
&= (-1)^{J_1 - 2j_1} [2j_{12}+1] \mathscr{F}_{\{j_{12}, j_{12}, 0\}}(\{\bar{u}, \bar{v}, 0\}) \mathscr{F}_{\{j_1, j_2, j_{12}\}}(\{x_1, x_2, u\}) \\
&\quad \times \mathscr{F}_{\{j_{12}, j_3, j_4\}}(\{v, x_3, x_4\}) \\
&= (-1)^{J_1 - 2j_1} [2j_{12}+1]^{1/2} [(2j_{12})!]^{-1} \rho(j_1 j_2 j_{12}) \rho(j_{12} j_3 j_4) \\
&\quad \times [\bar{u}\bar{v}]^{2j_{12}} [x_1 x_2]^{J_1 - 2j_{12}} [x_1 u]^{J_1 - 2j_2} [x_2 u]^{J_1 - 2j_1} \\
&\quad \times [v x_3]^{J_2 - 2j_4} [v x_4]^{J_2 - 2j_3} [x_3 x_4]^{J_2 - 2j_{12}}
\end{aligned} \tag{3.2}
$$

where $J_1 = j_1 + j_2 + j_{12}$, $J_2 = j_3 + j_4 + j_{12}$. Performing the differentions in (3.2) with the aid of the differential calculus in algebrae of SU(n) vector invariants (Karassiov *et al* 1979) we obtain this GI in an expanded form:

$$
\begin{aligned}
\mathscr{F}_{\{j_i\}}^{j_{12}}(\{x_i\}) &= [2j_{12}+1]^{1/2} \rho(j_1 j_2 j_{12}) \rho(j_{12} j_3 j_4) (-1)^{J_1 - 2j_2} [x_1 x_2]^{J_1 - 2j_{12}} [x_3 x_4]^{J_2 - 2j_{12}} \\
&\quad \times (J_1 - 2j_1)!(J_2 - 2j_3)!(J_1 - 2j_2)!(J_2 - 2j_4)! \\
&\quad \times \sum_{\alpha + \beta = J_2 - 2j_4} \frac{[x_1 x_3]^\alpha [x_2 x_3]^\beta [x_1 x_4]^{J_1 - 2j_2 - \alpha}}{\alpha! \beta! (J_1 - 2j_2 - \alpha)!} \frac{[x_2 x_4]^{J_1 - 2j_1 - \beta}}{(J_1 - 2j_1 - \beta)!} \\
&= (-1)^{J_2 + 2j_3} \left(\frac{(2j_{12}+1)!}{(J_1+1)!(J_2+1)!} \right)^{1/2} \frac{[x_1 x_2]^{J_1 - 2j_{12}} [x_3 x_4]^{J_2 - 2j_{12}}}{[(J_1 - 2j_{12})!(J_2 - 2j_{12})!]^{1/2}} \\
&\quad \times \sum_{\nu_1 + \nu_2 = j_1 - j_2} \left(\begin{matrix} L_1 \\ \nu_1 \end{matrix} \middle| \begin{matrix} L_2 \\ \nu_2 \end{matrix} \middle\| \begin{matrix} j_{12} \\ j_1 - j_2 \end{matrix} \right) \frac{[x_1 x_3]^{L_1 + \nu_1} [x_2 x_3]^{L_1 - \nu_1} [x_1 x_4]^{L_2 + \nu_2} [x_2 x_4]^{L_2 - \nu_2}}{[\Pi_{i=1}^2 (L_i + \nu_i)!(L_i - \nu_i)!]^{1/2}}
\end{aligned} \tag{3.3}
$$

where $\left(\begin{smallmatrix} L_1 \\ \nu_1 \end{smallmatrix} \middle| \begin{smallmatrix} L_2 \\ \nu_2 \end{smallmatrix} \middle\| \begin{smallmatrix} j_{12} \\ j_1 - j_2 \end{smallmatrix} \right)$ are second rank Clebsch-Gordan coefficients of SU(2) corresponding to the reduction $D(2L_1) \times D(2L_2) \to D(2L_1 + 2L_2)$, $L_1 = J_2/2 - j_4$, $L_2 = J_2/2 - j_3$. From (3.3) it follows that the matrix $S = \| S(\begin{smallmatrix} \{a_{ik}\} \\ \{j_i\}, j_{12} \end{smallmatrix}) \|$ of the transformation from the Kumar GI $\mathscr{F}_{\{a_{ik}\}} = \Pi_{i<k} [x_i x_k]^{a_{ik}}$ to the orthonomal GI $\mathscr{F}_{\{j_i\}}^{j_{12}}(\{x_i\})$ is given in terms of Clebsch-Gordan coefficients rather than 6-j symbols as is the case for a transformation matrix between two orthonormal sets of GI corresponding to two different standard

coupling schemes. Note that the matrix S^{-1} of the transformation inverse to (3.3) is also expressed in terms of SU(2) Clebsch-Gordan coefficients

$$\mathscr{F}_{\{a_{ik}\}}(\{x_i\}) = \prod_{i<k} [x_i x_k]^{a_{ik}} = \sum_{j_{12}} S^{-1}\begin{pmatrix} \{a_{ik}\} \\ \{j_i\}, j_{12} \end{pmatrix} \mathscr{F}^{j_{12}}_{\{j_i\}}(\{x_i\})$$

$$2j_i = \sum_{k\neq i} a_{ik} \qquad a_{ik} = a_{ki} \text{ if } k<i$$

$$\begin{aligned} S^{-1}\begin{pmatrix} \{a_{ik}\} \\ \{j_i\}, j_{12} \end{pmatrix} &\equiv \mathscr{F}^{j_{12}}_{\{j_i\}}(\{\bar{x}_i\})\mathscr{F}_{\{a_{ik}\}}(\{x_i\}) \\ &= \prod_{i=1}^{4} \delta_{2j_i, \Sigma_{k\neq i} a_{ik}} (-1)^{J_1-2j_2}[2j_{12}+1]^{1/2} \\ &\quad \times \prod_{i=1}^{2} a_{i_3}!\,a_{i_4}!\,(J_1+1)!\,(J_2+1)!\,(J_1-2j_1)!\,(J_1-2j_2)! \\ &\quad \times (J_1-2j_{12})!\,(J_2-2j_3)!\,(J_2-2j_4)!\,(J_2-2j_{12})! \\ &\quad \times \rho(j_1 j_2 j_{12})\rho(j_{12} j_3 j_4)[(J_2+1-a_{34})!]^{-1} \\ &\quad \times \sum_{\lambda} (-1)^{\lambda+J_2-2j_{12}-a_{34}}[\lambda!\,(a_{13}-\lambda)!\,(a_{24}-\lambda)!\,(J_2-a_{13}-2j_4+\lambda)! \\ &\quad \times (J_2-2j_3-a_{24}+\lambda)!\,(J_2-2j_{12}-a_{34}-\lambda)!]^{-1} \\ &= (-1)^{2j_3+a_{34}}\left(\frac{a_{13}!\,a_{23}!\,a_{14}!\,a_{24}!\,(J_1+1)!\,(J_2+1)!\,(J_1-2j_{12})!\,(J_2-2j_{12})!}{(J_2+1-a_{34})!\,(J_2-2j_{12}-a_{34})!}\right)^{1/2} \\ &\quad \times \left(\begin{matrix} K_1 \\ \mu_1 \end{matrix}\middle|\begin{matrix} K_2 \\ \mu_2 \end{matrix}\middle\|\begin{matrix} j_{12} \\ j_2-j_1 \end{matrix}\right) \end{aligned} \tag{3.4}$$

$$K_1 = j_3 - a_{34}/2 \qquad K_2 = j_4 - a_{34}/2 \qquad \mu_1 = \tfrac{1}{2}(a_{23}-a_{13}) \qquad \mu_2 = \tfrac{1}{2}(a_{24}-a_{14}).$$

Hence we obtain an expression for 6-j symbols in the form of a weighted sum of only two Clebsch-Gordan coefficients. Indeed, performing the change $x_1 \leftrightarrow x_3$ and $j_1 \leftrightarrow j_3$, $j_{12} \to j_{23}$ in (3.3) we obtain the expanded form of the GI $\mathscr{F}^{((23)(14)),j_{23}}_{\{j_i\}}(\{x_i\})$ for the coupling scheme ((23)(14)). Then, taking into account (3.4) and defining (for 6-j symbols) the formula

$$\mathscr{F}^{((23)(14)),j_{23}}_{\{j_i\}}(\{x_i\}) = \sum_{j_{12}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} [(2j_{12}+1)(2j_{23}+1)]^{1/2}(-1)^{J_1-2j_{12}} \mathscr{F}^{j_{12}}_{\{j_i\}}(\{x_i\}) \tag{3.5}$$

we find immediately from (3.3) and (3.4) that

$$\begin{aligned} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} &= (-1)^{J_1-2j_{12}}[(2j_{12}+1)(2j_{23}+1)]^{-1/2}\mathscr{F}^{j_{12}}_{\{j_i\}}(\{\bar{x}_i\})\mathscr{F}^{((23)(14)),j_{23}}_{\{j_i\}}(\{x_i\}) \\ &= (-1)^{J_1-2j_4}\left(\frac{(J_1+1)!\,(J_2+1)!\,(2j_{23})!\,(J_1-2j_{12})!\,(J_2-2j_{12})!}{(J_3+1)!\,(J_4+1)!\,(2j_{12}+1)}\right)^{1/2} \\ &\quad \times \sum_{\nu_1+\nu_2=j_2-j_3} \left(\begin{matrix} \tilde{L}_1 \\ \nu_1 \end{matrix}\middle|\begin{matrix} \tilde{L}_2 \\ \nu_2 \end{matrix}\middle\|\begin{matrix} j_{23} \\ j_2-j_3 \end{matrix}\right) \\ &\quad \times \left(\begin{matrix} j_3-\frac{1}{2}(\tilde{L}_2-\nu_2) \\ \frac{1}{2}(J_3-2j_{23}-\tilde{L}_1+\nu_1) \end{matrix}\middle|\begin{matrix} j_4-\frac{1}{2}(\tilde{L}_2-\nu_2) \\ \frac{1}{2}(\tilde{L}_2+\nu_2-J_4+2j_{23}) \end{matrix}\middle\|\begin{matrix} j_{12} \\ j_2-j_1 \end{matrix}\right)(-1)^{\tilde{L}_2-\nu_2} \\ &\quad \times [(\tilde{L}_1+\nu_1)!\,(\tilde{L}_2-\nu_2)!\,(J_2+1-\tilde{L}_2+\nu_2)!\,(J_2-2j_{12}-\tilde{L}_2+\nu_2)!]^{-1/2} \\ &\quad \tilde{L}_1 = J_4/2 - j_4 \qquad \tilde{L}_2 = J_4/2 - j_1 \\ &\quad J_3 = j_2+j_3+j_{23} \qquad J_4 = j_1+j_4+j_{23}. \end{aligned} \tag{3.6}$$

As another example we give the concise form of the GI $\mathscr{F}^{((12)(34)(56)),\{j_{12},j_{34},j_{56}\}}_{\{j_i\}}(\{x_i\})$ in which we use the operator $\mathscr{F}_{\{j_{12}j_{34}j_{56}\}}(\{\bar{y}_i\})$

$$\mathscr{F}^{((12)(34)(56)),\{j_{12}j_{34},j_{56}\}}_{\{j_i\}}(\{x_i\})$$

$$\begin{aligned}
&\equiv \sum_{\{m_i\}} \begin{pmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & -m_{12} \end{pmatrix}\begin{pmatrix} j_3 & j_4 & j_{34} \\ m_3 & m_4 & -m_{34} \end{pmatrix}\begin{pmatrix} j_5 & j_6 & j_{56} \\ m_5 & m_6 & -m_{56} \end{pmatrix} \\
&\quad\times \begin{pmatrix} j_{12} & j_{34} & j_{56} \\ m_{12} & m_{34} & m_{56} \end{pmatrix}[(2j_{12}+1)(2j_{34}+1)(2j_{56}+1)]^{1/2}(-1)^{j_1+j_3+j_5-j_2-j_4-j_6} \\
&\quad\times \prod_{i=1}^{6} \frac{(x_i^1)^{j_i+m_i}(x_i^2)^{j_i-m_i}}{[(j_i+m_i)(j_i-m_i)]^{1/2}} \\
&=(-1)^{\varphi}[(2j_{12}+1)(2j_{34}+1)(2j_{56}+1)]^{1/2} \\
&\quad\times \mathscr{F}_{\{j_{12}j_{34}j_{56}\}}(\{\bar{y}_i\})\mathscr{F}_{\{j_1j_2j_{12}\}}(\{x_1,x_2,y_1\}) \\
&\quad\times \mathscr{F}_{\{j_3j_4j_{34}\}}(\{x_3,x_4,y_2\})\mathscr{F}_{\{j_5j_6j_{56}\}}(\{x_5,x_6,y_3\}) \\
&=\rho(j_{12}j_{34}j_{56})\rho(j_1j_2j_{12})\rho(j_3j_4j_{34})\rho(j_5j_6j_{56})(-1)^{\varphi}[\bar{y}_1\bar{y}_2]^{J_2'-2j_{56}} \\
&\quad\times[\bar{y}_1\bar{y}_3]^{J_2'-2j_{34}}[\bar{y}_2\bar{y}_3]^{J_2'-2j_{12}}[x_1x_2]^{J_1'-2j_{12}}[x_1y_1]^{J_1'-2j_2}[x_2y_1]^{J_1'-2j_1} \\
&\quad\times[x_3x_4]^{J_1''-2j_{34}}[x_3y_2]^{J_1''-2j_4}[x_4y_2]^{J_1''-2j_3}[x_5x_6]^{J_1'''-2j_{56}} \\
&\quad\times[x_5y_3]^{J_1'''-2j_6}[x_6y_3]^{J_1'''-2j_5}[(2j_{12}+1)(2j_{34}+1)(2j_{56}+1)]^{1/2}
\end{aligned}$$

$$\varphi=j_{12}+j_{34}+j_{56}+j_1+j_3+j_5-j_2-j_4-j_6 \qquad J_1'=j_1+j_2+j_{12}$$

$$J_1''=j_3+j_4+j_{34} \qquad J_1'''=j_5+j_6+j_{56} \qquad J_2'=j_{12}+j_{34}+j_{56}. \tag{3.7}$$

Now we discuss the results obtained in the light of the discussion of the previous section. In accordance with it the above algorithm gives recurrence procedures for constructing orthonormal bases of irreps $D(OJ\dot{O}_{m-3})$ of SU(m), $m>2$, with all possible subgroup chains like (2.8) and (2.10) and therefore the sets $\{j_1,\ldots,j_m\}$, α label these bases completely. In particular, the algorithm provides a realisation of the Gel'fand-Tsetlin bases $\{e^J_\mu\}$ of irreps $D(OJ\dot{O}_{m-3})$ which is different from other known forms (see, e.g., Nagel and Moshinsky 1965, Wu 1971, Fujiwara and Horiuchi 1982). Therefore a linear connection exists between $\{j_i\}$, α and parameters m_{ij} of the Gel'fand-Tsetlin pattern $\mu=[m_{ij}]$. For example, the GI from (3.2) and (3.3) give concise and expanded forms, respectively, for vectors of the Gel'fand-Tsetlin basis of SU(4) irrep $D(OJO)$. Therefore

$$2j_i=\sum_{k=1}^{i} m_{ki}-\sum_{k=1}^{i-1} m_{ki-1} \qquad 2j_{12}=m_{12}-m_{22} \qquad J=\sum_{i=1}^{4} j_i \tag{3.8}$$

(cf Klink (1983) where Gel'fand-Tsetlin pattern of SU(n) irreps are used for labelling SU(2) coupling vectors within another approach).

Note that our technique also allows us to construct bases of SU(m) irreps $D(J-a, a, \dot{O}_{m-3})$ in terms of SU(2) GI. Indeed, fixing the SU(2) vector x_{m+1} in GI, which correspond to vectors of the SU($m+1$) irrep $D(OJ\dot{O}_{m-2})$, we obtain vectors of the SU(m) irrep $D(J-a, a, \dot{O}_{m-3})$. Specifically, setting $x_4=e_i\equiv(\delta_i^k)$, $i=1, 2$, in (3.2), we obtain some vectors of the SU(3) irrep $D(J-a, a)$ (see also Karassiov and Shelepin 1968).

Furthermore, the above results allow us to reinterpret other results of the theory of angular momenta in terms of irreps $D(OJ\dot{O}_{m-3})$ of higher groups SU(m).

Specifically, the mth rank Wigner coefficients of SU(2) may be considered as second rank Clebsch-Gordan coefficients of SU(m) for the reduction $D(J\dot{O}_{m-2}) \times D(J\dot{O}_{m-2}) \to D(OJ\dot{O}_{m-3})$ and $3n$-j symbols of SU(2) may be treated as matrix elements of a transformation between the bases, belonging to different types, of groups SU(m). We emphasise that such an outlook which originates, as a matter of fact, from Regge's work (Regge 1958) is useful both for constructing a unified theory of SU(m) symmetries and determining various properties of particular quantities (cf Karassiov *et al* 1979, Ališauskas 1983, Fujiwara and Horiuchi 1982).

4. Conclusion

Unlike the previous case $n=2$ we do not have any explicit analytical expression for the orthonormal GI $\mathscr{F}^{\gamma}_{\{P_1P_2P_3\}}(\{x_i|i=1,2,\ldots,3n-3\})$ for the third rank Wigner coefficients corresponding to the reduction

$$D(P_1)\times D(P_2)\times D(P_3)\to D(\dot{O}_{n-1}) \tag{4.1}$$

with any admissible signatures $P_i=(p_1^i, p_2^i, \ldots, p_{n-1}^i)$ of irreps $D(P_i)$ (the label γ distinguishes equivalent irreps $D(\bar{P}_3)$ in the Clebsch-Gordan series $D(P_1)\times D(P_2)=\Sigma_{P_3}\sigma(P_3)D(\bar{P}_3)$, $\bar{P}_3=(p_{n-1}^3,\ldots,p_2^3,p_1^3)$), whose bases are realised in terms of $(n-1)$ vectors x_i. Therefore we cannot extend the recurrence procedure of the previous section for constructing both orthonormal GI of SU(n), $n\geq 3$, and polynomial bases of $D(\dot{O}_{n-1}J\dot{O}_{m-n-1})$ of SU(m) with any standard coupling schemes or subgroup chains.

However, we have obtained explicit analytical expressions for the GI $\mathscr{F}_{\{P_1,P_2,P_3\}}(\{x_i\})$ when one signature in (4.1) has the form $P_i=(p_1^i\dot{O}_{n-2})$ (Karassiov and Shelepin 1980). This result allows us to obtain a polynomial realisation for the Gel'fand-Tsetlin bases of irreps $D(\dot{O}_{n-1}J\dot{O}_{m-n-1})$ of SU(m) and the corresponding GI of SU(n) in the form (2.3).

As for the GI for the reduction (4.1) we may proceed as follows. Using the identification of these GI with basis vectors of the U($3n-3$) irrep $D[\dot{J}_n\dot{O}_{2n-3}]$ for the non-canonical subgroup chain $\mathrm{U}(3n-3)\supset[\mathrm{U}(n-1)]^{\otimes 3}$ we may construct them as linear combinations of the Gel'fand-Tsetlin basic vectors (in the above polynomial realisation). For determining the coefficients of these expansions we have a set of algebraic equations which follow from the conditions that GI $\mathscr{F}^{\gamma}_{\{P_i\}}(\{x_i\})$ should be eigenvectors of the $[\mathrm{U}(n-1)]^{\otimes 3}$ Casimir operators and of extra operators Γ_i with the eigenvalues γ_i, $\gamma=(\gamma_i)$ (see Racah 1964) which may be taken as mutually commuting $\mathrm{S}[\mathrm{U}(n-1)]^{\otimes 3}$ scalars in the enveloping algebra of the group SU($3n-3$) (or in a larger operator set). (We note that it is sufficient to determine GI for standard realisation of bases of irreps $D(P_i)$ in terms of $(n-1)$ vectors x_i when these GI are the highest vectors with respect to subgroup $\mathrm{S}[\mathrm{U}(n-1)]^{\otimes 3}$ of SU($3n-3$).) An example of such a procedure has been given by us (Karassiov and Shchelock 1986) for the case $n=3$. Our analysis has shown that many conditions of this sort reduce to fixing some numbers m_{ij} in the Gel'fand-Tsetlin pattern $\mu=[m_{ij}]$ of the appropriate vectors (which establishes interesting interrelations between bases of different types) whereas non-trivial equations are generated only by the Casimir operators of one subgroups $\mathrm{SU}(n-1)\subset\mathrm{U}(n-1)$ and the operators Γ_i. We still need further study to obtain an optimal calculational scheme for realising this idea in general cases; the main problems to be resolved here are: a suitable choice of the operators Γ_i, determination of the spectra $\{\gamma_i\}$ and an explicit construction of operators shifting the eigenvalues γ_i (cf Van der Jeugt *et al*

1983). One can also use an alternative scheme for finding GI $\mathcal{F}^{\gamma}_{\{p_i\}}(\{x_i\})$ based on explicitly constructing the orthonormal set of the Wigner operators in the GI-like form with the aid of the operators x_i^{α}, $\bar{x}_j^{\beta}$ (cf Louck 1979, Biedenharn and Flath 1984).

In conclusion we emphasise that the analysis carried out above gives new interesting possibilities concerning a way of developing a unified mathematical technique of SU(n) symmetries and of other classical groups and supergroups (cf Ol'shanskii and Prati 1985, Howe 1976).

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